7. The wave equation in one space variable

The partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

describes the vibration of a uniform elastic string, the oscillation of air columns in musical instruments, the propagation of electrical signals along uniform wires. In each of these cases $u(x, t)$ represents respectively at the position x and time t, the displacement from rest of the stretched string, the excess air pressure and the voltage or current. For obvious reasons this equation is called the *one-dimensional wave equation* . The characteristic equation is

$$\frac{dt}{dx} = \pm \frac{1}{c} .$$

The two families of characteristics are $x \pm ct$ = constant. The change of variable

$\xi = x + ct$, $\eta = x - ct$, leads to the normal form

$$u_{\xi\eta} = 0.$$

Integrating with respect to η

$$u_{\xi} = f(\xi),$$

f an arbitrary function.

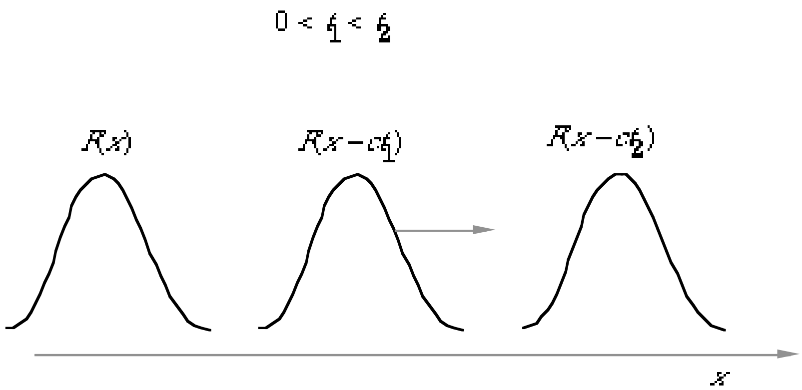
Integrating again with respect to ξ

$$u(\xi, \eta) = \int f(\xi)d\xi + G(\eta) = F(\xi) + G(\eta),$$

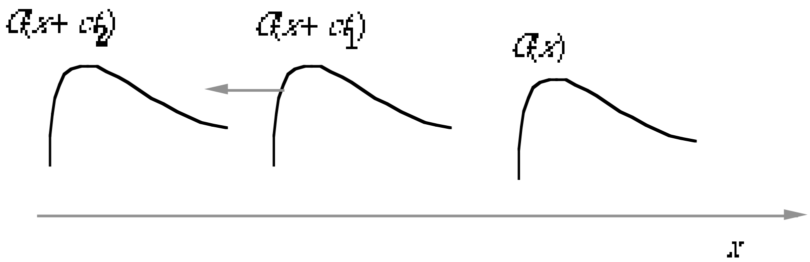
F, G arbitrary functions . Therefore the general solution to the one-dimensional wave equation is

$$u(x, t) = F(x - ct) + G(x + ct),$$

F, G arbitrary twice differentiable functions. $F(x - ct)$ represents a disturbance travelling in the direction x increasing as t increases



and $G(x + ct)$ represents a disturbance travelling in the direction x decreasing as t increases,



We see that the general solution of the homogeneous one-dimensional wave equation is the sum of forward and backward travelling waves. The speed of the wave motion is c.